Erdos-Faber-Lovasz Conjecture

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Abstract

A conjecture of Erdos, Faber, and Lovasz states that if p complete graphs, each having exactly p vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with p colors. This conjecture can be broken up into three cases, (i) graphs that contain no propellers, (ii) graphs with propellers where the pins are non adjacent, (iii) graphs with propellers and adjacent pins. We will prove the affirmative for (i) and provide a bound of $p + 1$ for (ii).

1 Introduction

Paul Erdos, born in Hungary circa 1913, grew to be one of the most famous mathematicians. He was known for his uncanny abilities as a problem solver as much as he was for living an eccentric lifestyle. In his lifetime he published around 1,500 mathematical aticles[1]. The only other mathematician who can even begin to compare with Erdos's accomplishments is Leonhard Euler. Part of the key to Erdos's success, in addition to being a naturally gifted thinker, was his highly collaborative nature. He believed that mathematics was more of a social activity, and by the time he passed away had 511 different collaborators[2]. Not only did he collaborate with a lot of people, but he worked in so many different branches of mathematics that eventually people created what is called the Erdos number. A person's Erdos number represents said person's degree of separation from Erdos himself based upon their collaboration with him. So a person who worked directly with Erdos on a paper would have an Erdos number of 1, while a person who worked on a paper with that collaborator would have an Erdos number of 2. Most mathematicians have an Erdos number less than 5, which is a true testament to how prevalent Erdos was within the mathematical community.

However, despite Erdos's great contributions to mathematics, he himself was not so much a theory builder as much as he was a problem solver. Erdos loved to solve problems and throughout his career would offer payments for solutions to unsolved problems. His payments would range from $25 for problems that he felt were just outside the reach of current mathematical thinking to several thousand dollars for problems that were both extremely difficult along with being mathematically significant. For example, one of the most notable of these problems is the Erdos conjecture on arithmetic progressions which Erdos offered $5,000 for a solution[3].

Vance Faber is an American born mathematician who specializes in combinatorics, applied linear algebra, and image processing. He received his Ph.D. from Washington University in Saint Louis and continued on to become a professor at the University of Colorado at Denver[4]. Throughout the years that followed he worked at Los Alamos National Laboratory as a part of the Computer Research and Applications group, and eventually became the Group Leader.

Laszo Lovasz, another prominent Hungarian mathematician, is best known for his work in combinatorics. He has received many awards for his contributions to the mathematical community including the Wolf Prize, Knuth Prize and the Kyoto Prize[5]. Lovasz has worked at many prestigious universities including Eotvos Lorand University in Hungary along with Yale University. Additionally, in 2014 Lovasz was elected to be the president of the Hungarian Academy of Sciences.

The Erdos-Faber-Lovasz conjecture was initially valued by Erdos at $50 but several years had passed without any solution. Erdos recognized the surprising difficulty of the problem and raised the reward to $500. This means that one of the world's greatest mathematicians to ever live viewed this to be a very challenging problem. We state the conjecture of Erdos, Faber, and Lovasz as follows.

Conjecture If p complete graphs, each having exactly p vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with p colors.

2 Background Information

Definition A *graph* G is a finite, nonempty set of vertices together with a (possibly empty) set of unordered pairs of distinct vertices called edges.

Definition The *degree of a vertex* v in a graph G is the number of edges of G incident with v.

Definition Δ will be used to represent the maximum degree in a graph G.

Definition δ will be used to represent the minimum degree in a graph G.

Definition A graph G is *regular of degree* r if for each vertex v of G, deg $v = r$.

Definition A *complete (p,q) graph* is a regular graph of degree $p-1$, we denote this graph by K_p.

Definition A graph H is a *subgraph* of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

Definition We say H is an *induced subgraph* of G if $H \subset G$ and $E(H) = \{uv|u, v \in V(H) \text{ and } uv \in E(G)\}$

Definition An assignment of colors to the vertices of a graph G, one color to each vertex, so that adjacent vertices are assigned different colors is called a *coloring* of G.

Definition A coloring in which n colors are used is an *n-coloring*. The minimum n for which a graph G is *n-colorable* is called the *chromatic number* of G and is denoted by $\chi(G)$.

3 Examples

Terminology: We call a graph an *EFL-graph* if it satisfies the conditions of the Erdos-Faber-Lovasz Conjecture.

To gain a better understanding of what the conjecture is stating we will look at a few examples of EFL-graphs.

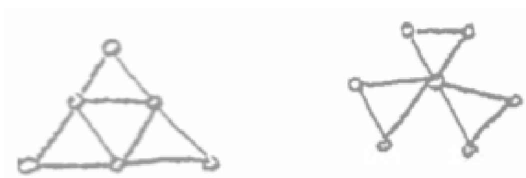


Figure 1: p=3

Figure 1 shows two different EFL-graphs with $p = 3$. Notice that in each case, there are three copies of K_3, with each pair of complete graphs only overlapping in one vertex, and that we can quite easily color these graphs in three colors.

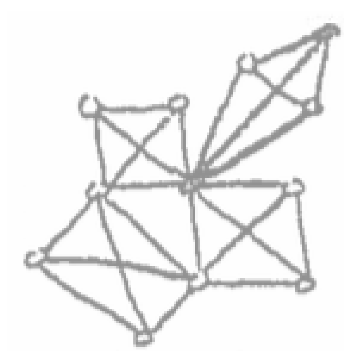
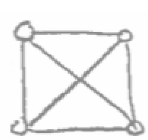


Figure 2: p=4

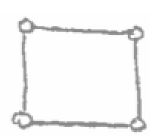
Figure 2 shows an EFL-graph with $p = 4$. Again notice that there are four copies of K_4, with each pair of complete graphs only overlapping in one vertex, and this graph can also be colored using four colors.

Figure 3: p=5

Figure 3 shows an EFL-graph with $p = 5$. It is hard to see with all the edges. To make it easier to see these graphs we will show only the outer cycle of each complete graph. So for example K_4 normally looks like this



but now with our new notation it will look like this



So now we can redraw Figure 3 as follows to make it easier to visualize.

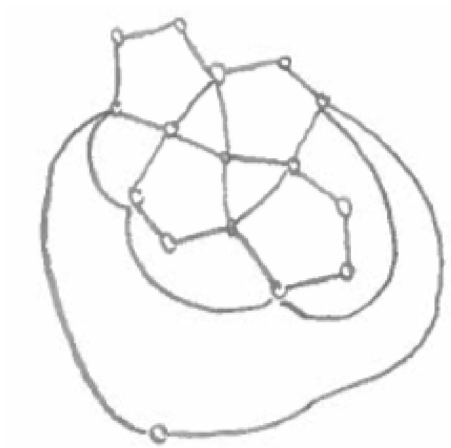


Figure 4: p=5 with new notation

For the remainder of the paper we will use this shorthand notation to represent complete graphs.

4 Partial Results

This conjecture was first formulated in 1972, and has since has been reformulated into equivalent versions relating to both hypergraphs and even combinatorial arguments in hopes of making progress. In 1988, using hypergraphs, Chiang and Lawler showed that if G is an EFL-graph,

$$\chi(G) \leq \frac{3p}{2} - 2.$$

In 1992, Kahn improved this result to show that for a nearly-disjoint hypergraph on n vertices which is an EFL-graph

$$\chi(G) \leq p + o(p).$$

The most recent result came from Romero and Alonso-Pecina in September of 2014. They showed that the conjecture is true for all

$$p \leq 12.$$

Why this problem still remains unsolved is a mystery, as it is easy to comprehend, yet deceptively difficult to prove[6][7][8].

5 General Coloring Theorems

While graph coloring is a relatively new area of mathematics there are a few general theorems that have been developed which give an upper bound on the coloring number for a general graph. However, in this section we will discuss why these theorems have little use to us in making progress on this conjecture.

Theorem 5.1. First General Theorem on Vertex Coloring*: For every graph G, $\chi(G) \leq 1 + \Delta(G)$. This states that the coloring number of a graph G is less than or equal to one more than the maximum degree of the graph.*

This theorem is of little use to us because in an EFL graph, any two complete graphs sharing a common vertex will then have a vertex with a degree of at least $2(p-1)$, see figure below, which by this theorem would then give us an upper bound of $2(p-1)+1$ for the coloring number, which is much higher than our target for the conjecture.

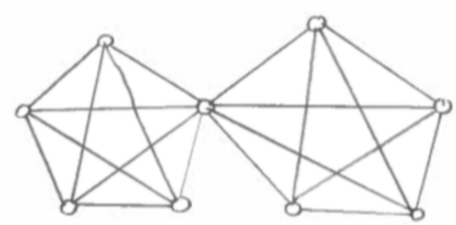


Figure 5: Two K_5 that share a single vertex

Theorem 5.2. Second General Theorem on Vertex Coloring*: For every graph G, $\chi(G) \leq 1 + max\ \delta(G')$, where the maximum is taken over all induced subgraphs G' of G. In other words, the coloring number of a graph G is less than or equal to one more than the maximum of the minimum degrees on all induced subgraphs.*

It is a little harder to see where this theorem fails. However if we look at the case when $p = 5$ and we have the property that every complete graph is connected to every other complete graph, then we will note that there are five vertices with degree 4 and the remaining all have degree 8. So it would seem that the theorem would hold because our $\delta = 4$ and thus $\delta + 1 = 5$. However, we must keep in mind that we need to look at the maximum of the minimum degrees on all induced subgraphs.

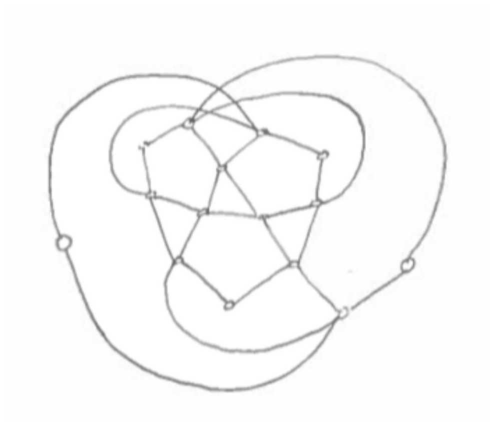


Figure 6: EFL-graph with p=5 and each K_5 is connected to every other K_5

Let us consider the induced subgraph in which we remove all of the vertices of degree 4. Now we have obtained a new graph (as shown below) which is 6-regular.

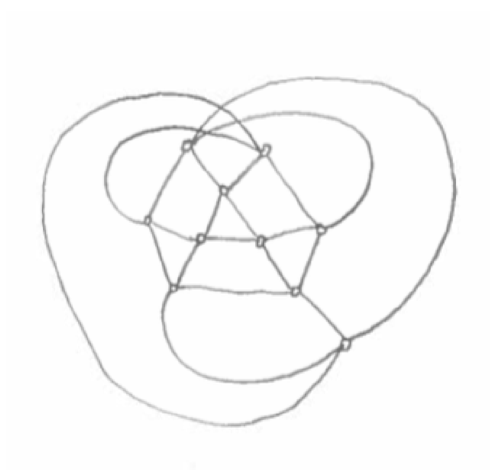


Figure 7: The induced subgraph formed by removing all vertices of degree 4

Hence the minimum degree of this graph is 6 which by the above theorem would imply that the graph is $6 + 1 = 7$ colorable. This is an improvement over the first theorem, however, this is only one configuration and the fact that it did not work means that not only can we not use it to solve this conjecture, but it also means that we cannot use it to form any helpful upper bounds.

6 Maximum Connected Graphs

In order to make further progress on this conjecture we must first introduce some new definitions. Let it also be noted that throughout the remainder of the paper let the graph G denote a graph which satisfies the conditions in the conjecture.

Definition A graph G is called *maximum connected* if every complete graph $K_{p_i} \subset G$, shares a vertex with every other complete graph in G.

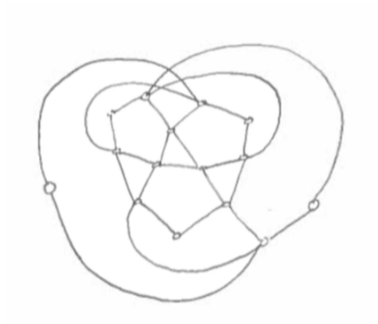


Figure 8: Maximum Connected EFL-graph when p=5

Definition Given a graph G the *skeletal subgraph* is the induced subgraph of G where the vertices are all of those that are shared with more than one copy of a complete graph.

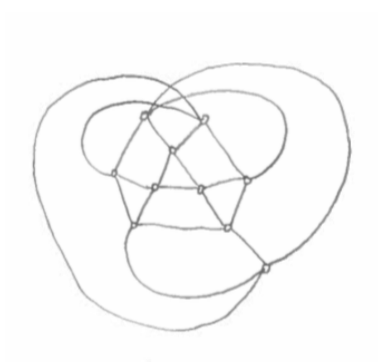


Figure 9: Skeletal Subgraph when p=5

Definition A *propeller* is the union of three or more complete graphs that all share a single vertex.

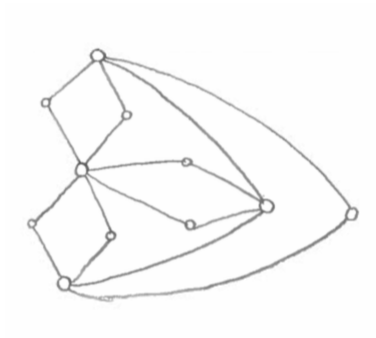


Figure 10: An EFL-graph with a propeller when p=4

In this section we will show that all graphs G that don't have any propellers have a coloring number of p.

Lemma 6.1. *Let H be the skeletal subgraph of a graph G. If $\chi(H) \leq p$ then $\chi(G) = p$.*

Proof.
Assume H is the skeletal subgraph of a graph G and $\chi(H) \leq p$. By the construction of the skeletal subgraph the vertices not included in H are only included in one complete graph. Therefore, these vertices are connected to exactly $p-1$ vertices. We also know that the skeletal subgraph contains at most $p-1$ vertices of a single K_p graph. Therefore, if the skeletal subgraph is p colorable, then a copy of a complete graph in G can have at most $p-1$ colored vertices from the skeletal subgraph. This leaves at least one color left to color the remaining vertex. Hence, the entire graph is p colorable. $\square$

Now we will look at the skeletal subgraphs of maximum connected graphs and note the following.

- There are $\sum_{i=0}^{p-1}(p-1) - i = \frac{p^2-p}{2}$ vertices.
- Since every vertex is shared between two copies of K_{p-1}, we have that every vertex has degree $2(p-2)$.
- There are p copies of K_{p-1} embedded within the subgraph, because initially there were p copies of K_p but since we removed one vertex from each individual copy we now have copies of K_{p-1}.
- Every copy of K_{p-1} shares a unique vertex with every other copy of K_{p-1}, and this is by virtue of the fact that there are no propellers.

Lemma 6.2. *Let* $n = \frac{p^2-p}{2}$*, and let* H *be the skeletal subgraph of a maximum connected graph. Then,*

$$\chi(H) = \frac{n}{\lfloor \frac{p}{2} \rfloor}$$

Proof.
Proof by Induction, Base Case: Consider when $p = 4$

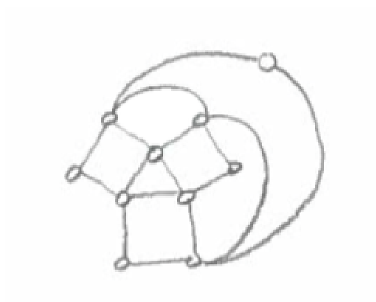


Figure 11: Maximum connected case when p=4

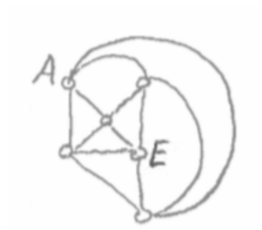


Figure 12: Skeletal Subgraph of Maximum Connected Graph when p=4

WLOG pick a vertex A. A is included in two copies of K_{p-1} and is adjacent to two vertices in the remaining copies of K_{p-1}. Removing vertex A and all of the vertices adjacent to A eliminates two copies of K_{p-1} and two vertices from both of the remaining graphs. Thus leaving only one vertex E. Therefore, A and E can be colored the same color, moreover, the entire subgraph can be colored in 3 colors and $3 = \frac{6}{\lfloor \frac{4}{2} \rfloor}$

Case 2: $p = 5$

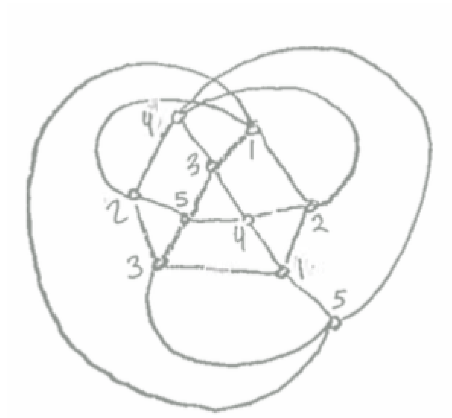


Figure 13: Skeletal Subgraph of Maximum Connected Graph when p=5

The proof for this is the same as the proof for when $p = 4$ however we now note that the subgraph can be colored using only 5 colors and $5 = \frac{10}{\lfloor \frac{5}{2} \rfloor}$

Inductive Step: Suppose this holds true for all $p \leq k$. Consider the case where $p = k + 1$. This implies that the skeletal subgraph has $k + 1$ copies of K_k. If we pick one copy say K_{k_1}, and let v be a vertex in K_{k_1}, then we know that v is also a vertex in another complete graph which we will call K_{k_2}. (We know this because every vertex in the skeletal subgraph is defined because it is shared between two copies of K_{k+1}). Similarly for that same reason, each vertex in K_{k_1} that is not v is also a vertex in a unique other copy of K_{k_i}, and the same is true for the vertices in K_{k_2}. Therefore, if we remove v and all of the vertices that are adjacent to v we end up removing all of K_{k_1} and K_{k_2}. This in turn removes two vertices from all the remaining complete graphs. Hence, we now have a subgraph with $k - 1$ copies of K_{k-2}. By our inductive hypothesis we have that this subgraph H_{k-1} satisfies the following, $\chi(H_{k-1}) = \frac{n_{k-1}}{\lfloor \frac{k-1}{2} \rfloor}$, where $n_{k-1} = \frac{(k-1)^2-(k-1)}{2}$. However, since we already picked a vertex v before creating this subgraph we have that $\chi(H_{k+1}) = \frac{n_{k-1}}{\lfloor \frac{k-1}{2} \rfloor} + 1 = \frac{n_{k+1}}{\lfloor \frac{k+1}{2} \rfloor}$. □

Corollary 6.3. *All maximum connected graphs are p colorable.*

Proof. By lemma 2.2 we know that the coloring number of the skeletal subgraph of a maximum connected graph can be given by $\chi(H) = \frac{n}{\lfloor \frac{p}{2} \rfloor}$. It is important to note that this is equal to $p - 1$ when p is even and p when p is odd. Therefore, $\chi(H) \leq p$. Hence by lemma 2.1, this implies that $\chi(G) = p$. □

Theorem 6.4. *If a connected graph G contains no propellers then $\chi(G) = p$.*

Proof. Let G be a connected graph with no propellers and G^* the corresponding maximum connected graph with no propellers. Similarly, let H be the skeletal subgraph of G, and H^* be the skeletal subgraph of G^*. Then we have that $H \subset H^*$. By lemma 2.2 we have that $\chi(H^*) \leq p$ and since $H \subset H^*$ it follows that $\chi(H) \leq p$. Since H is the skeletal subgraph of G, we have that $\chi(G) = p$ by lemma 2.1. $\square$

7 Graphs with Propellers

Definition A *propeller* is the union of three or more complete graphs that all share a single vertex.

Definition The vertex that is shared within the propeller is called the *pin*.

Definition Each complete graph that is a part of the propeller is called a *wing*.

We have just shown that all graphs that do not contain a propeller satisfy the conditions of the conjecture. In this portion we will show that if a graph contains propellers and the pins are not adjacent, then

$$\chi(G) \leq \begin{cases} p, & \text{if } p \text{ is even} \\ p+1, & \text{if } p \text{ is odd} \end{cases}$$

First we will cover the trivial cases. If a graph G has a single propeller with p wings then it is trivially p colorable. If a graph G has a single propeller with $p-1$ wings and a complete graph sharing a vertex with each wing then the skeletal subgraph is isomorphic to K_p. Therefore the skeletal subgraph is p colorable so the graph G is p colorable.

7.1 Inversions

We will now describe the process in which we create what is called the *inversion* of a graph.

If there are w wings then there will always be at least $w-1$ vertices not connected to any other copy of a complete graph. Therefore, if we look at each wing and take out the nonshared vertices and the pin we will always be able to get w copies of K_w (because each wing will have a copy of the pin included). Now if we remove the pin from all of the graphs we will obtain w copies of K_{w-1}. We can arrange these into the maximum connected graph. Inserting this into the original graph yields a graph with no propellers.

Example 7.1.
Constructing the inversion on a graph with $p = 4$.

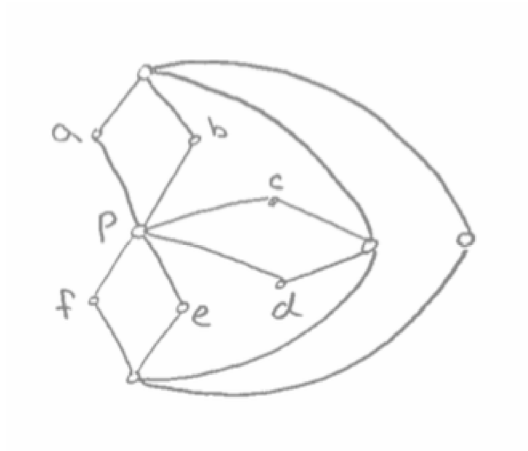


Figure 14: An EFL graph with one propeller when p=4

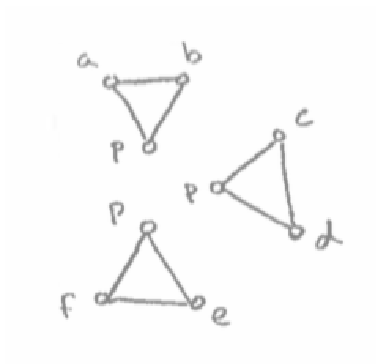


Figure 15: Isolate the unshared vertices and the pin

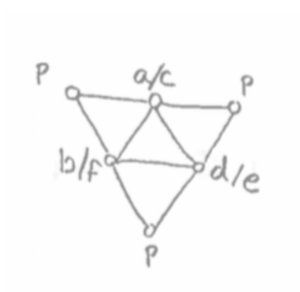


Figure 16: Union graphs into maximum connected case such that the pins are never shared between two complete graphs

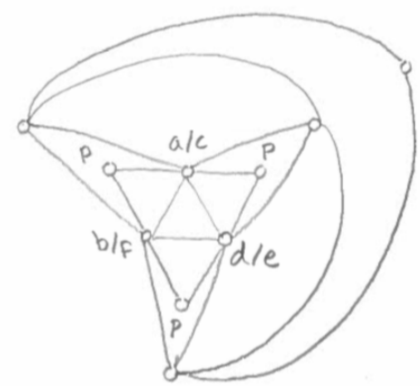


Figure 17: Insert the above figure back into the original graph

Theorem 7.2. *If a graph G has a propeller then*

$$\chi(G) \leq \begin{cases} p, & \textit{if } p \textit{ is even} \\ p+1, & \textit{if } p \textit{ is odd} \end{cases}$$

Proof.
Suppose a graph G has a propeller. Then we can do an inversion on the pin to create a maximum connected graph in which the pin is not shared in any of the copies of the complete graphs formed. Inserting the inversion back into the original graph G will produce a new graph G' which contains no propellers. Therefore, by lemma 2.2 we know the skeletal subgraph H of G' satisfies the following,

$$\chi(H) \leq \begin{cases} p-1, & \text{if } p \text{ is even} \\ p, & \text{if } p \text{ is odd} \end{cases}$$

Hence, if p is even we can then color the pin with the p^{th} color, and if p is odd then we will need an extra color to color the pin so we would need $p+1$ colors. It then follows that we have

$$\chi(G) \leq \begin{cases} p, & \text{if } p \text{ is even} \\ p+1, & \text{if } p \text{ is odd} \end{cases}$$

□

Corollary 7.3. *If a graph G contains more than one propeller, and no two pins are adjacent, then*

$$\chi(G) \leq \begin{cases} p, & \textit{if } p \textit{ is even} \\ p+1, & \textit{if } p \textit{ is odd} \end{cases}$$

Proof.
Assume a graph G has more than one propeller and no two pins are adjacent. Then we can do an inversion around every pin in the graph G. Now we have a graph G' which contains no propellers. As in Theorem 3.2 we know that this implies that $\chi(G') \leq p$ implying that we will need a new color for each pin. But since no two pins are adjacent we can use this newly introduced color to color all the pins. Therefore,

$$\chi(G) \leq \begin{cases} p, & \text{if } p \text{ is even} \\ p+1, & \text{if } p \text{ is odd} \end{cases}$$

□

8 Bibliography

[1] Hoffman. The Man Who Loved Only Numbers: The Story of Paul Erdos and the Search for Mathematical Truth. New York: Hyperion, 1998. Print.
[2] Erdos Number Project Data. Web. 7 May 2015.
[3] Soifer. The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of its Creators. Print.
[4] "Mathematics Genealogy Project." The. Web. 7 May 2015.
[5] Lovasz. "Curriculum Vitae." Web.
[6] Chiang, and Lawler. "Edge Coloring of Hypergraphs and a Conjecture of Erdos, Faber, Lovasz." Print.
[7] Kahn. "Coloring Nearly-disjoint Hypergraphs with n+o(n) Colors." Print.
[8] Romero, and Alonso-Pecina. "The Erdos-Faber-Lovasz conjecture is true for $n \leq 12$." Print.